

# Universal amplitude ratios in the two-dimensional Ising model<sup>1</sup>

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## Abstract

We use the results of integrable field theory to determine the universal amplitude ratios in the two-dimensional Ising model. In particular, the exact values of the ratios involving amplitudes computed at nonzero magnetic field are provided.

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<sup>1</sup>Work supported by the European Union under contract FMRX-CT96-0012

Universality is one of the most fascinating concepts of statistical mechanics [1]. Briefly stated, it says that physical systems with different microscopic structure but having in common some basic internal symmetry exhibit the same critical behaviour in the vicinity of a phase transition point. The point is best illustrated considering the singular behaviour of the various thermodynamic quantities nearby the critical point. For a magnetic system exhibiting a second order phase transition the usual notation is

$$\begin{aligned}
C &\simeq (A/\alpha) \tau^{-\alpha}, & \tau > 0, \quad h = 0 \\
C &\simeq (A'/\alpha') (-\tau)^{-\alpha'}, & \tau < 0, \quad h = 0 \\
C &\simeq (A_c/\alpha_c) |h|^{-\alpha_c}, & \tau = 0, \quad h \neq 0 \\
M &\simeq B (-\tau)^\beta, & \tau < 0, \quad h = 0^+ \\
\chi &\simeq \Gamma \tau^{-\gamma}, & \tau > 0, \quad h = 0 \\
\chi &\simeq \Gamma' (-\tau)^{-\gamma'}, & \tau < 0, \quad h = 0 \\
\chi &\simeq \Gamma_c |h|^{-\gamma_c}, & \tau = 0, \quad h \neq 0 \\
h &\simeq D_c M |M|^{\delta-1}, & \tau = 0, \quad h \neq 0 \\
\xi &\simeq \xi_0 \tau^{-\nu}, & \tau > 0, \quad h = 0 \\
\xi &\simeq \xi'_0 (-\tau)^{-\nu'}, & \tau < 0, \quad h = 0 \\
\xi &\simeq \xi_c |h|^{-\nu_c}, & \tau = 0, \quad h \neq 0
\end{aligned}$$

where  $\tau = a(T - T_c)$  ( $a$  positive constant),  $h$  is the applied magnetic field and the limit towards the critical point  $\tau = 0, h = 0$  is understood.  $C, M, \chi$  and  $\xi$  denote the specific heat, the magnetisation, the susceptibility and the correlation length, respectively.

The critical exponents  $\alpha, \beta, \dots$  are the same for all systems within a given universality class and are related by the scaling and hyperscaling relations in  $d$  dimensions

$$\begin{aligned}
\alpha &= \alpha', \quad \gamma = \gamma', \quad \nu = \nu', \\
\gamma &= \beta(\delta - 1), \quad \alpha = 2 - 2\beta - \gamma, \quad 2 - \alpha = d\nu, \\
\alpha_c &= \alpha/\beta\delta, \quad \gamma_c = 1 - 1/\delta, \quad \nu_c = \nu/\beta\delta.
\end{aligned}$$

The critical amplitudes  $A, B, \dots$ , on the other hand, depend on the scale factors used for  $\tau$  and  $h$  and are nonuniversal. However, universal ratios of amplitudes can be constructed in which any dependence on metric factors cancels out. Together with the critical exponents, these ratios further characterise the given universality class. The standard amplitude combinations considered in the literature are [2]

$$A/A', \quad \Gamma/\Gamma', \quad \xi_0/\xi'_0, \tag{1}$$

$$R_C = A\Gamma/B^2, \quad R_\xi^+ = A^{1/d}\xi_0, \tag{2}$$

$$R_\chi = \Gamma D_c B^{\delta-1}, \quad R_A = A_c D_c^{-(1+\alpha_c)} B^{-2/\beta}, \quad Q_2 = (\Gamma/\Gamma_c)(\xi_c/\xi_0)^{\gamma/\nu}. \tag{3}$$

A substantial progress was made over the last years in the derivation of nonperturbative theoretical results in two-dimensional statistical mechanics and quantum field theory. The solution of conformal field theories (CFTs) [3, 4] provided an almost complete classification of universality classes for second order phase transitions in  $d = 2$ . In particular, it solved the problem of the exact determination of the critical exponents. The critical amplitudes, however, carry information about the scaling region outside the critical point and are not determined by CFT. In this respect, the Zamolodchikov's observation that specific perturbations of the critical point lead to integrable off-critical theories [5] is of crucial importance. The integrable theories obtained in this way, regarded as quantum field theories in  $1 + 1$  dimensions, are characterisable through the determination of their exact  $S$ -matrix. A number of physical quantities can then be computed using different techniques [6]. In particular, the results provided by the thermodynamic Bethe ansatz (TBA) [7] and the form factor approach [8, 9, 10, 11] enable the determination of amplitude ratios and have been used for this purpose in the problems of self-avoiding walks [12] and percolation [13] (see also [14]).

It is the purpose of this note to illustrate the derivation of universal amplitude ratios from (integrable) field theory through the very basic example of the two-dimensional Ising model. Of course, the purely "thermal" ratios (1) and (2) are exactly known for this case since the seventies, when the correlation functions of the Ising model at  $h = 0$  were first computed on the lattice [15, 16]. The possibility to determine the ratios (3), on the contrary, relies on the more recent realisation that the scaling limit of the Ising model at  $\tau = 0$  and  $h \neq 0$  is an integrable theory [5].

We will regard the scaling limit of the two-dimensional Ising model as described by the euclidean field theory defined by the action

$$\mathcal{S} = \mathcal{S}_{CFT} - \tau \int d^2x \varepsilon(x) - h \int d^2x \sigma(x) , \quad (4)$$

where  $\mathcal{S}_{CFT}$  stays for the Ising critical point conformal action and  $\sigma$  and  $\varepsilon$  denote the magnetisation and energy operators, respectively. All the critical exponents are determined by the scaling dimensions  $x_\sigma = 1/8$  and  $x_\varepsilon = 1$  of the operators  $\sigma$  and  $\varepsilon$ :  $\alpha = (d - 2x_\varepsilon)/(d - x_\varepsilon) = 0$ ,  $\beta = x_\sigma/(d - x_\varepsilon) = 1/8$ ,  $\gamma = 7/4$ ,  $\delta = 15$ ,  $\nu = 1$ ,  $\alpha_c = 0$ ,  $\gamma_c = 14/15$ ,  $\nu_c = 8/15$ . The physical dimensions of the two couplings in (4) are  $\tau \sim m^{2-x_\varepsilon}$  and  $h \sim m^{2-x_\sigma}$ ,  $m$  being a mass parameter. What is important for us is that the theory (4) becomes integrable when at least one of the two couplings is set to zero [5].

Choosing the scale factors for  $\tau$  and  $h$  amounts to fixing a normalisation for the conjugated operators  $\varepsilon$  and  $\sigma$ . We will proceed to determine the critical amplitudes within the standard CFT normalisation defined by the asymptotic conditions

$$\langle \sigma(x) \sigma(0) \rangle = |x|^{-1/4} , \quad x \rightarrow 0$$

$$\langle \varepsilon(x) \varepsilon(0) \rangle = |x|^{-2}, \quad x \rightarrow 0. \quad (5)$$

Let us begin with the amplitudes for the correlation length. The latter can be defined in different ways. For the time being, we will refer to the so called “true” correlation length defined through the large distance decay of the spin-spin correlation function

$$\langle \sigma(x) \sigma(0) \rangle^c \sim \frac{e^{-|x|/\xi}}{|x|^{(d-1)/2}}, \quad |x| \rightarrow \infty \quad (6)$$

where  $\langle \cdots \rangle^c$  denotes the connected correlator. From the representation of the correlator as a spectral sum over  $n$ -particle intermediate states

$$\langle \sigma(x) \sigma(0) \rangle^c = \sum_n |\langle 0 | \sigma(0) | n \rangle|^2 e^{-E_n |x|}, \quad (7)$$

it is clear that the “true” correlation length is the inverse of the total mass of the lightest state entering the decomposition above. For integrable models, the exact relation between the particle masses and the couplings appearing in the action is provided by the TBA [17]. The result for the Ising model is<sup>1</sup>

$$\begin{aligned} m_\tau &= 2\pi |\tau|, \\ m_h &= \mathcal{C} |h|^{8/15}, \\ \mathcal{C} &= \frac{4 \sin \frac{\pi}{5} \Gamma(1/5)}{\Gamma(2/3) \Gamma(8/15)} \left( \frac{4\pi^2 \Gamma(3/4) \Gamma^2(13/16)}{\Gamma(1/4) \Gamma^2(3/16)} \right)^{4/15} = 4.40490857\dots \end{aligned} \quad (8)$$

Due to the invariance under spin reversal at  $h = 0$ , the magnetisation operator couples only to the states with odd (even) number of particles for  $\tau > 0$  ( $\tau < 0$ ). When  $h \neq 0$ ,  $\sigma$  couples to any intermediate state. It follows

$$\begin{aligned} \xi_0 &= 2\xi'_0 = 1/(2\pi), \\ \xi_c &= 1/\mathcal{C}. \end{aligned} \quad (9)$$

The specific heat diverges logarithmically in the Ising model ( $\alpha = 0$ ) and the specific heat amplitudes are accordingly defined through  $C \simeq -A \ln \tau$  and analogous relations for  $A'$  and  $A_c$ . Writing the partition function as  $\text{Tr} \exp(-\mathcal{S})$  and the reduced free energy as  $f = -1/V \ln Z$ , the specific heat per unit volume is given by

$$C = -\frac{\partial^2 f}{\partial \tau^2} = \int d^2x \langle \varepsilon(x) \varepsilon(0) \rangle^c \sim -2\pi \ln m r_0, \quad (10)$$

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<sup>1</sup>We denote by the subscript  $\tau$  ( $h$ ) the quantities referring to the theory (4) with  $h = 0$  ( $\tau = 0$ ).  $m_h$  is the mass of the lightest among the 8 particles the mass spectrum of the  $\tau = 0$  theory consists of.

where (5) was used and  $r_0$  is an ultraviolet cutoff. Recalling the relations between the mass scale  $m$  and the coupling constants  $\tau$  and  $h$ , the specific heat amplitudes in the CFT normalisation are simply

$$A = A' = 2\pi, \quad A_c = 2\pi \frac{8}{15}. \quad (11)$$

The magnetisation per unit volume is given by

$$M = -\frac{\partial f}{\partial h} = \langle \sigma \rangle. \quad (12)$$

Vacuum expectation values in the CFT normalisation are exactly known for integrable models due to the TBA [17] and some more recent developments [18]. For the magnetisation operator in the Ising model at  $\tau < 0, h = 0$  and at  $\tau = 0, h > 0$  they are, respectively<sup>2</sup>

$$\begin{aligned} \langle \sigma \rangle_\tau &= B (-\tau)^{1/8}, \\ \langle \sigma \rangle_h &= \frac{2\mathcal{C}^2}{15 (\sin \frac{2\pi}{3} + \sin \frac{2\pi}{5} + \sin \frac{\pi}{15})} h^{1/15} = 1.27758227.. h^{1/15}, \end{aligned} \quad (13)$$

with

$$B = 2^{1/12} (2\pi/e)^{1/8} \mathcal{A}^{3/2} = 1.70852190..; \quad (14)$$

we used the Glaisher's constant

$$\mathcal{A} = 2^{7/36} \pi^{-1/6} \exp \left[ \frac{1}{3} + \frac{2}{3} \int_0^{1/2} dx \ln \Gamma(1+x) \right] = 1.28242712.. \quad (15)$$

From (13) we deduce

$$D_c = \left[ \frac{15 (\sin \frac{2\pi}{3} + \sin \frac{2\pi}{5} + \sin \frac{\pi}{15})}{2\mathcal{C}^2} \right]^{15} = 0.0253610264.. \quad (16)$$

The reduced susceptibility is defined as

$$\chi = -\frac{\partial^2 f}{\partial h^2} = \int d^2x \langle \sigma(x) \sigma(0) \rangle^c. \quad (17)$$

The amplitudes  $\Gamma$  and  $\Gamma'$  were computed exactly in [16] integrating the  $h = 0$  spin-spin correlator. In the CFT normalisation they read<sup>3</sup>

$$\begin{aligned} \Gamma &= 0.148001214.., \\ \Gamma' &= 0.00392642280.. \end{aligned} \quad (18)$$

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<sup>2</sup>In the following we will consider for convenience only positive values of  $h$ . Due to the symmetry about  $h = 0$  this involves no loss of generality.

<sup>3</sup>The relations one needs to pass from the lattice normalisation of [16] to the one used here are  $\sigma_{lat} = 2^{5/48} e^{1/8} \mathcal{A}^{-3/2} \sigma$  and  $[(T - T_c)/T_c]_{lat} = (\pi / \ln(1 + \sqrt{2})) \tau$ .

When  $\tau = 0$ ,  $\sigma$  is the operator which perturbs the critical point. The zeroth moment of the spin-spin correlator is then related to the vacuum expectation value as

$$\int d^2x \langle \sigma(x) \sigma(0) \rangle_h^c = \frac{x_\sigma}{(2 - x_\sigma)h} \langle \sigma \rangle_h . \quad (19)$$

It follows

$$\Gamma_c = \frac{1}{\delta} D_c^{-1/\delta} = 0.0851721517.. . \quad (20)$$

The dependence on the operator normalisations (the only nonuniversal ingredient we had in our computation) cancels when the combinations (1), (2) and (3) are considered. The exact results we obtain for the ratios are collected in Table 1. The values for the combinations of purely thermal amplitudes are well known. Concerning the ratios involving amplitudes computed at  $h \neq 0$ , to the best of our knowledge the only reliable estimates in  $d = 2$  come from the series analysis of Ref. [19]. The value  $R_\chi \simeq 6.78$  is quoted there together with a value for  $Q_2$  which uses the “second moment” correlation length (see below). No previous result for  $R_A$  is known to us.

We conclude this note computing the ratios which involve the correlation length amplitudes using the second moment correlation length

$$\xi_{2nd}^2 \equiv \frac{1}{2d} \frac{\int d^2x |x|^2 \langle \sigma(x) \sigma(0) \rangle_h^c}{\int d^2x \langle \sigma(x) \sigma(0) \rangle_h^c} . \quad (21)$$

The amplitudes  $(\xi_0)_{2nd}$   $(\xi'_0)_{2nd}$  could be exactly computed by numerical integration of the spin-spin correlation function at  $h = 0$ . Here, we will take a short cut with the purpose of illustrating a point which is particularly relevant for more general applications. In integrable models the spectral sum (7) is not a purely formal expression because the matrix elements it contains (known as form factors) can be computed exactly. While the Ising model at  $h = 0$  remains the single example for which the resummation of the spectral series is known, a remarkably fast convergence of the series emerged in the last years as a general feature of integrable models [10, 12, 20]. This important circumstance makes the form factor approach extremely effective for obtaining accurate quantitative results at a relatively little cost.

We compute  $(\xi_0)_{2nd}$  and  $(\xi'_0)_{2nd}$  in the leading form factor approximation (1-particle contribution for  $\tau > 0$ , 2-particle contribution for  $\tau < 0$ ). Using<sup>4</sup> [21]

$$\begin{aligned} \langle 0 | \sigma(0) | \theta \rangle_\tau &= \langle \sigma \rangle_\tau , \\ \langle 0 | \sigma(0) | \theta_1, \theta_2 \rangle_\tau &= i \langle \sigma \rangle_\tau \tanh \frac{\theta_1 - \theta_2}{2} , \end{aligned} \quad (22)$$

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<sup>4</sup>On-shell momenta are parameterised as  $p^\mu = (m_\tau \cosh \theta, m_\tau \sinh \theta)$ .

one finds

$$\begin{aligned}(\xi_0)_{2nd} &\simeq 1/(2\pi), \\ (\xi'_0)_{2nd} &\simeq 1/(2\sqrt{10}\pi) .\end{aligned}\tag{23}$$

$(\xi_c)_{2nd}$  could be similarly estimated using the form factors for the  $\tau = 0$  model computed in [20]. However, its exact value is immediately determined reminding that  $\sigma$  is the perturbing operator at  $\tau = 0$ . Then the  $c$ -theorem sum rule holds [22]

$$c = \frac{3}{4\pi} [2\pi h(2 - x_\sigma)]^2 \int d^2x |x|^2 \langle \sigma(x) \sigma(0) \rangle_h^c, \tag{24}$$

where the central charge  $c$  equals  $1/2$  for the Ising universality class. Combining with (19) one gets

$$(\xi_c)_{2nd} = \sqrt{\frac{8}{45\pi}} D_c^{1/30} = 0.21045990.. \tag{25}$$

Table 2 contains the results one obtains for the amplitudes ratios. The value  $(\xi_0/\xi'_0)_{2nd} = 3.16..$  is quoted in Ref. [23] as the result of the integration of the exact correlator;  $(Q_2)_{2nd} = 2.88 \pm 0.02$  is the series result obtained in Ref. [19].

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## Table Caption

**Table 1** . Exact amplitude ratios for the two-dimensional Ising model. The results for  $\xi_0/\xi'_0$ ,  $R_\xi^+$  and  $Q_2$  refer to the “true” correlation length.

**Table 2** . Results referring to the “second moment” correlation length with  $(\xi_0)_{2nd}$  and  $(\xi'_0)_{2nd}$  computed in the leading form factor approximation.

$A/A' = 1$
$\Gamma/\Gamma' = 37.6936520..$
$\xi_0/\xi'_0 = 2$
$R_C = 0.318569391..$
$R_\xi^+ = 1/\sqrt{2\pi}$
$R_\chi = 6.77828502..$
$R_A = 0.0250658794..$
$Q_2 = 3.23513834..$

**Table 1**

$(\xi_0/\xi'_0)_{2nd} \simeq 3.162$
$(R_\xi^+)_{2nd} \simeq 0.3989$
$(Q_2)_{2nd} \simeq 2.833$

**Table 2**